International Journal of Theoretical Physics, Vol. 44, No. 2, February 2005 (© 2005)
DOI: 10.1007/s10773-005-1491-5

Quantization as Selection Problem

Peter Enders[1] and Dieter Suisky[2]

Received November 18, 2004; accepted November 24, 2004

Quantum systems exhibit a smaller number of energetic states than classical systems (A. Einstein, 1907, Die Plancksche Theorie der Strahlung und die Theorie der spezifischen Wärme, *Ann. Phys.* **22**, 180ff). We take up the selection criterion for this in two parts. (1) The selection problem between classical and nonclassical mechanical systems is formulated in terms of possible and impossible configurations (among others, this overcomes the difficulties occurring when discussing the behavior of quantum particles in terms of paths). (2) The (nonclassical) selection of the quantum states is formulated, using recurrence relations and the energy law. The reformulation of "quantization as eigenvalue problem" in terms of "quantization as selection problem" allows one to derive Schrödinger's stationary equation from classical mechanics through a straightforward and unique procedure; the nonstationary and multibody equations are subsequently acquired within the same frame. In contrast to the (classical) eigenvalue problem, the (nonclassical) selection problem can be formulated and solved without any reference to additional a priori assumptions on the nature of the quantum system, such as the wave-corpuscle dualism or an underlying wave equation or the existence of Planck's finite action parameter. The existence of such an additional parameter—as the only additional one—is inherent in the procedure. Within our axiomatic-deductive approach, we modify classical mechanics only where it itself indicates an inherent limitation.

KEY WORDS: quantization; Schrödinger equation.

1. INTRODUCTION

From the very beginning (Planck, 1900), the development of quantum physics is indivisibly connected with the discussion of its relationship to classical physics. And even after the overwhelming success and general acceptance of quantum theory, it had to be stated, that "It is in principle impossible, however, to formulate the basic concepts of quantum mechanics without using classical mechanics" (Landau and Lifschitz, 1959). As a consequence, the foundation of quantum mechanics (QM) can be done in two fundamentally different ways.

[1] Fischerinsel 2, D-10179 Berlin, Germany; e-mail: enders@dekasges.de.
[2] Institute of Physics, Humboldt University Berlin, Newtonstr. 15, D-12489 Berlin, Germany; e-mail: dsuisky@physik.hu-berlin.de.

Way 1 Taking classical mechanics (CM) as necessary, but not sufficient and, consequently, needing *additional* assumptions, such as
- to restrict the energy spectrum to the values $n \cdot h\nu$ (Einstein, 1907; Planck, 1900) or to $\frac{n}{2}h\nu$ (Bohr, 1913),
- to "*distinguish*" (Heisenberg, 1977) or to "*select*" (Messiah, 1999; Pauli, 1926) the values $n \cdot \hbar$ of the action integral $\oint pdq$ (n—integer; in contrast to CM, the action integral is *not* subject to a variational principle),
- to suppose the existence of h and to abandon the classical paths (Heisenberg, 1925),
- to suppose the existence of h and of a wave function being the solution of an eigenvalue problem (Schrödinger, 1927).

Way 2 Alternatively, taking CM as necessary *and* sufficient, i.e., supposing, that such additional assumptions are *not* necessary (Enders and Suisky, Einführung in die Quantenfeldtheorie des Festkörpers, manuscript in preparation; Suisky and Enders, 2001). Speeding ahead, we note, that according to Bohr's main prepositions (Bohr, 1913), one has to use for this a representation of CM, where only the state conservation, but *not* the state change is axiomatically fixed. Such representations have been developed by Euler (Euler, III–1) and by Helmholtz (1911). Furthermore, Bohr's and Schrödinger's theories show, that *Helmholtz's rule*, that a system changes its extension when exchanging work with the environment, holds in a certain sense within QM, too. And within both Bohr's and Schrödinger's theories, energy and extension are *internal* system parameters.

Thus, this contribution is organized according to the following key prereqisite for succeeding along Way 2.

- Euler's representation of CM allows for formulating well-defined relations between classical and nonclassical equations of motion, since only the principles of state *conservation* are fixed axiomatically, while—in contrast to Newton's axiomatics—that of state change are not.
- Helmholtz's treatment of the energy conservation law induces a selection problem between possible and impossible configurations of classical systems.
- Euler's method of maxima and minima can be generalized, in order to derive a nonclassical representation of the energy law, the stationary Schrödinger equation being a condition for its validity.
- Einstein's criterion for the difference between classical bodies and atomic aggregates (the latter ones should exhibit less stationary states than those) can be reformulated in terms of Whittaker's integral representation of solutions of differential equations, in order to solve the stationary Schrödinger equation as a selection problem rather than as an eigenvalue problem.

- Euler's principles of state conservation and of state change of classical bodies can be generalized to classical conservative systems and be reformulated in terms of quantum-mechanical entities, in order to derive the time-dependent Schrödinger equation.

Moreover, we will sketch how this approach can be exploited for *deriving* two fundamental properties of multibody quantum systems, viz., the permutation symmetry of the wavefunction and the indistinguishability of identical particles. We will outline, how this approach justifies the method of field quantization through normal-mode expansion, where the quantization concerns only the temporal, but not the spatial dependence of the field variable. In the Summary, we will also discuss Schrödinger's analysis of the relationship between CM and QM.

2. THE MODIFICATION OF NEWTON'S AXIOMATICS BY EULER

2.1. Euler's Treatment of Classical Mechanics

Leonhard Euler (Euler, 1911, III–1, III–11) was the first who applied the calculus to *all* areas of mathematics and mechanics of his time, and he developed new areas. Moreover, he worked out an axiomatics of mechanics, where only Newton's 1st axiom concerning the conservation of state is retained as an axiom, while Newton's 2nd and 3rd axioms concerning the change of state are treated as problems to be solved. This allows for introducing *alternative* equations of motion *without* loosing the contact to CM. (It also resembles the removal of Euklid's 5th axiom leading to non-Euklidean geometries). The *logical* structure of Euler's axiomatics suggests to formulate these alternatives as *selection problems*.

We remark, that

- Newton's and Euler's notion of state corresponds to the nowadays notion of *stationary* state;
- D'Alembert (1743) and Maupertuis (1768) also have modified and developed Newton's axiomatics;
- originally, the quantum conditions were called *state* conditions ("Zustandsbedingungen," see Pauli, 1926).

Euler has developed a unified concept of bodies and forces. There is only one type of bodies and only one type of forces. The conservation of state is not due to a force, but due to the very nature of the bodies. The latter is given through their general properties. These are extension, movability, inertia and impenetrability. The impenetrability is the fundamental, "*essential*" property, from which the other three properties can be derived.

The forces appear only due to the competition for space occupation. Thus, the interaction between bodies is primarily that of elastic collisions, where forces are created in just that amount, which is necessary for preventing the penetration of one body into another body. Hence, the magnitude of the force (and of the action it performs) is *minimum*.

We remark, that the elastic collision represents the only genuine classical-mechanical interaction. It exhibits no interaction constant; this is another reason for the fact, that CM became that protophysics, on which methodologically all other physical disciplines are built.

2.2. Euler's Axioms—Equations of State

The motion of bodies is described in terms of state conservation and state changes. The existence of (stationary) states is postulated in the following axioms.

Axiom E0 Every body is *either* resting *or* moving. This means, that the subsequent axioms E1 and E2 are not independent of each other; they exclude each other and, at once, they are in *harmony* with each other (Euler, II–5b).

Axiom E1 A body preserves its state at rest, unless an external cause sets it in motion.

Axiom E2 A body preserves its state of straight uniform motion, unless an external cause forces it to change this state.

Let us quote three observations, which justify the separation of the state at rest ("rest" refers to a given reference system, of course).

- According to Malebranche (1906), "nothing" (no motion, no speed) is not the limit of "something existing" (motion, finite amount of speed).
- Dirac has argued for the existence of an absolute motion (Dirac, 1980).
- The distinction between rest and motion plays an important role in the transformation properties of the Hamilton–Jacobi equation (Faraggi and Matone, 1998; their path from CM to QM fails for the state at rest being absent within QM, indeed, cf. below).

The state variable is the velocity vector, $\mathbf{v}$ (for Euler, the mass of a body is always constant—the use of the velocity rather than the momentum as state variable may be caused by the fact, that it is closer to Leibniz's living force, mv^2). Thus, the *equation of state* reads $Z = \mathbf{v} = \mathbf{0}$ for the state at rest and $Z = \mathbf{v} = \text{const}$ for the state of straight uniform motion.

Note, that when one takes also the position, x, as state variable, one faces the problem, that the state is changed even in the absence of an external cause for doing so (Weizsäcker, 2002).

2.3. Equations-of-State-Change and Equations of Motion—A Selection Problem

The change of the state variable under the influence of an external cause is described through the *equation-of-state-change*. Here, it turns out to be crucial, that there are two quite different possibilities for the relationship between the state variable and its change.

Newton–Eulerian Assumption. The change of state is *independent* of the state itself. Thus, the equation-of-state-change reads $dZ = d\mathbf{v}(t) = \frac{1}{M}\mathbf{F}dt$. For constant mass, M, this leads to the Newtonian equation of motion, $M\ddot{\mathbf{x}} = \mathbf{F}$ [first published in (Euler, II–5a) in explicit calculus form].

Non-Newton–Eulerian Assumption. The change of state *does* depend on the state itself. In this case, there should exist a function $f(v(t))$ describing how $d\mathbf{v}$ depends on v; say, $d\,[f(v(t)) \cdot \mathbf{v}(t)] = \frac{1}{M}\mathbf{F}dt$. Since $f(v)$ is dimensionless, there is a reference velocity such, that $f(v) = f(v/v_{\text{ref}})$. Applying quite general requirements, the new function $f(v/v_{\text{ref}})$ can be calculated, and one arrives at *special-relativistic* relations, in particular, at the Lorentz transformation (Suisky and Enders, 2005). (Another derivation of the Lorentz transformation without referring to electrodynamics can be found in Mittelstaedt, 1995.)

Now, while Newton's 2nd law *fixes* the manner of state change to the first possibility, Euler's axioms allow to *choose* between both possibilities.

In other words, there is a *selection problem* between nonrelativistic ($d\mathbf{v}$ does *not* depend on $\mathbf{v}$) and relativistic CM ($d\mathbf{v}$ *does* depend on $\mathbf{v}$). It makes clear, which kind of modifications of the axiomatics of CM is necessary for obtaining non-Newtonian or even nonclassical theories.

3. CLASSICAL CONSERVATIVE SYSTEMS

The selection problem between CM and QM is much more sophisticated than that between relativistic and nonrelativistic mechanics, because the very foundation of CM is affected, viz., the general properties of bodies. For this, one has to resort to the more general *energetic* state description. Helmholtz (1911) has provided the strict foundation of the mechanical energy law sought for. He developed an ingenious method for the definition of new mechanical quantities through special combinations of mechanical concepts and logical relations between them. The lack of space forces us to make just some few references to Helmholtz's work. In order to concentrate onto the essentials, we assume, that the center of gravity of the system under consideration stays at the origin of the coordinate system and that the minimum of the potential energy of a conservative system equals zero.

3.1. Energy Law and Extension

The energy law of CM can be derived via direct integration of Newton's equation of motion,

$$M\ddot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)) \tag{1}$$

by means of its multiplication with $\dot{\mathbf{x}}$ (Euler, III–1, §§74–75).

$$M\ddot{\mathbf{x}} \cdot \dot{\mathbf{x}} = \frac{d}{dt}\left(\frac{M}{2}\dot{\mathbf{x}}^2\right) = \mathbf{F} \cdot \dot{\mathbf{x}} = \frac{d}{dt}\int^{\mathbf{x}} \mathbf{F} \cdot d\mathbf{x}' \tag{2}$$

provided that there is a function $\tilde{V}(\mathbf{x})$ such, that

$$\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F}(\mathbf{x}') \cdot d\mathbf{x}' = \tilde{V}(\mathbf{x}) - \tilde{V}(\mathbf{x}_0); \quad \mathbf{F}(\mathbf{x}) = \nabla\tilde{V}(\mathbf{x}) \tag{3}$$

Euler called this function "effectiveness" ("Wirksamkeit"). Then, Eq. (2) yields the conservation law (cf. also Euler, II-5b)

$$\frac{M}{2}\dot{\mathbf{x}}(t)^2 - \tilde{V}(\mathbf{x}(t)) = \text{const} = E \tag{4}$$

Helmholtz (1847) was the first who realized the universal relevance of this constant of integration, E. His more axiomatic approach (Helmholtz, 1911), however, starts with the conservation of "living forces" (we will use the modern notions).

Leibniz's law. The total kinetic energy, $T(\mathbf{v})$, of a system assumes the *same* value, when the system returns to the same configuration, C, (cf. Leibniz, 1982).

Helmholtz's conclusion. In this case, despite of being *defined* in terms of velocity, the kinetic energy is a pure function of the coordinates; the existence of such a function has to be required.

Then, $\frac{d}{dt}T(\mathbf{v}) = \mathbf{F} \cdot \dot{\mathbf{x}}$ is the total differential of a function depending soleily on the coordinates, and Eq. (2) can be written as, say,

$$\frac{d}{dt}T(\mathbf{v}) = \frac{d}{dt}\left[-V(\mathbf{x})\right] \tag{5}$$

provided, that $\mathbf{F} = -\nabla V(\mathbf{x})$ (the minus sign will be discussed below). The latter is a condition for the spatial arrangement of the forces within the system, viz., $curl\mathbf{F} \equiv \mathbf{0}$.

Upon simple integration, Eq. (5) becomes the *energy law* of CM,

$$V(\mathbf{x}) + T(\mathbf{v}) = E = \text{const} \tag{6}$$

Of course, within CM, the constancy of the l.h.s. is realized through the motion along a path curve: $\mathbf{x} = \mathbf{x}(t)$, $\mathbf{v}(t) = \dot{\mathbf{x}}(t)$, where the increase of the one term is compensated by the decrease of the other term.

We will call the stationary state with the value E of total energy shortly the "state E."

In Eq. (5), Helmholtz has set $-V(\mathbf{x})$ rather than $+\tilde{V}(\mathbf{x})$, in order to express through the potential energy function $V(\mathbf{x})$ the "disponible work storage" of the system. [The work, A, is not defined through the motion along trajectories according to the laws of motion, but through the slow displacement of a body, say, from $\mathbf{x}_1$ to $\mathbf{x}_2$, where the internal forces are balanced out by an external force. If there is a potential energy function, $V(\mathbf{x})$, then, $A = V(\mathbf{x}_2) - V(\mathbf{x}_1)$.]

Furthermore, $V(\mathbf{x})$ has the advantage over the function $\tilde{V}(\mathbf{x}) = -V(\mathbf{x})$ of more directly expressing

Helmholtz's rule. A system changes its extension in configuration space when exchanging work with the environment.

This rule can be generalized to the case of exchanging energy.

Generalized Helmholtz's rule. A system changes its extension in configuration space when exchanging energy with the environment.

For the sake of a more symmetrical treatment of space and momentum variables, we make the following *complementary* statements.

Complementary Leibniz's law. The energy of relative positions of a system, $V(\mathbf{x})$, is *un*changed, when it returns to the same momentum configuration, P.

Complementary Helmholtz's conclusion. Despite of being *defined* in terms of position, the potential energy can be written as a pure function of the momenta; the existence of such a function has to be required.

Taking for this function the expression $E - T(\mathbf{p})$, we reobtain the energy conservation law (6), where the velocities are replaced with the momenta. Choosing, again, the minus sign [this time, at the function $T(\mathbf{p})$], we deal with the kinetic energy function, $T(\mathbf{p})$, which, among others, has the advantage over $-T(\mathbf{p})$ of more directly expressing the following relationship between energy and extension in momentum space.

Complementary Generalized Helmholtz's rule. A system changes its extension in momentum space when exchanging energy with the environment.

Thus, generally speaking, the classical functions $V_{\mathrm{cl}}[C] = V(\mathbf{x})$ and $T_{\mathrm{cl}}[P] = T(\mathbf{p})$ provide *relationships between energy and extension* in space and momentum space, respectively. For the linear harmonic oscillator, these relationships read

$$E = V(x_{\max}) = \frac{\kappa}{2} x_{\max}^2 \quad \text{and} \quad E = T(p_{\max}) = \frac{1}{2M} p_{\max}^2 \tag{7}$$

The oscillator's extensions in space and momentum space are represented through $x_{\max}$ and $p_{\max}$, respectively.

These relationships induce *ordering* relations, which will proof to be useful when considering nonclassical systems, viz., for each ordered set $E_1 < E_2 < \cdots$ of a bounded system, there are ordered sets $x_{\max,1} < x_{\max,2} < \cdots$ and $p_{\max,1} < p_{\max,2} < \cdots$.

3.2. Possible and Impossible Configurations

Because of the validity of the equations of motion in *whole* space and to the freedom in the choice of the initial conditions, $\mathbf{x}_0$ and $\mathbf{p}_0$, a body can move to *any other* location (not being occupied by another body—in what follows, we will always understood, that Euler's ban is obeyed). For a system, this means, that *all* configurations are possible. (Each possible position can serve as initial condition; the phase space orbit is a sequence of possible initial conditions.) However, for a conservative system in a *fixed* state E, the initial (momentum) configurations, $\mathbf{x}_0$ and $\mathbf{p}_0$, are not independent of each other, but interrelated through the energy law as $T(\mathbf{p}_0) + V(\mathbf{x}_0) = E$. As a consequence, *not all* (momentum) configurations are possible in state E.

In fact, the inequality $T(\mathbf{p}_0) \geq 0$ implies the most important condition

$$V(\mathbf{x}_0) \leq E \tag{8}$$

It expresses the fact, that the work storage is never larger than the total energy. This fact implies the *limitation*, that *no* configuration, $\mathbf{x}$, can be assumed, where $V(\mathbf{x}) > E$. In other words, in each *fixed* state E, the set of *possible* configurations, $C_{\mathrm{cl}}^{\mathrm{poss}}$, does *not* comprise the *whole* configuration space, C^{all}, but is *bounded* such, that

$$C_{\mathrm{cl}}^{\mathrm{poss}} = C_{\mathrm{cl}}^{\mathrm{poss}}(E) = \{\mathbf{x} | V(\mathbf{x}) \leq E\} \subset C^{\mathrm{all}} = \{\mathbf{x}\} \tag{9}$$

For our model system, we have $|x(t)| \leq x_{\max}(E)$, cf. Eq. (7).

Hence, by virtue of the energy law, the configuration space, C^{all}, is divided into two disjunct domains, $C_{\mathrm{cl}}^{\mathrm{poss}}(E)$ and $C_{\mathrm{cl}}^{\mathrm{imposs}}(E) = C_{\mathrm{cl}}^{\mathrm{poss}}(E) \setminus C^{\mathrm{all}}$. An analogous division takes place for the set of all momentum configurations.

This *inherent limitation* of the motion classical-mechanical systems implies the following

Point of generalization (1). Under which conditions or for which mechanical systems this division into possible and impossible configurations takes *not* place, and which is the mechanics describing such systems?

3.3. Euler's Method of Maxima and Minima: Internal and External System Parameters

According to Einstein's criterion and in view of the *structural* stability of the electronic states in atoms, we need a representation of the energy not as a

continuous function of continuous external variables (such as the initial conditions), but as an *internal* system parameter.

Internal system parameters are the mass and other material parameters, but also the *extremalous* values of the *functions* $V(x)$ and $T(p)$, e.g., $V_{\min} = 0$ and $T_{\min} = 0$. According to Euler, these extremalous values can be obtained by means of the calculus, independent of the actual motion of the system. The minima of the potential energy function $V(x)$ define the stable equilibrium states of the system.

In contrast to these internal parameters, the initial conditions, x_0 and p_0 (we consider still the linear oscillator), are *external* parameters. All functions of them are *external* parameters as well, in particular, the total energy, $E = T(p_0) + V(x_0)$, and the maximum possible values of the potential and kinetic energies in state E, viz., $V_{\max}^{\text{poss}}(E) = V(x_{\max}) = E$ and $T_{\max}^{\text{poss}}(E) = T(p_{\max}) = E$, respectively. In contrast to the minimum possible values, $V_{\min}^{\text{poss}} = V_{\min} = 0$ and $T_{\min}^{\text{poss}} = T_{\min} = 0$, the values $V_{\max}^{\text{poss}}(E)$ and $T_{\max}^{\text{poss}}(E)$ are *not* extremalous values of the *functions* $V(x)$ and $T(p)$, respectively.

Thus, the actual (stationary) state of a system is determined by *both*, internal and external parameters.

There is one important exception, however. The state at rest is *uniquely* and determined through the condition

$$V = V_{\min} \quad \text{and} \quad T = T_{\min} \tag{10}$$

Here, *both* the potential and the kinetic energies assume *extremalous* values. This means, that the state at rest is *completely* governed by the "internal principles." In other words, we have

Claim 1. Parameters the values of which are determined by the condition $V = V_{\text{extrem}}$ and $T = T_{\text{extrem}}$ are, for this case, *internal* parameters.

As an example, we note that in the state at rest of the harmonic oscillator not only the minimum, but *also* the maximum extensions in space and in momentum space are determined by this condition and, thus, are *internal* parameters. For our model system, this refers to the state at rest with $x(t) \equiv 0$ and $p(t) \equiv 0$. This suggests

Claim 2. Sets of states, where the minimum *and* the maximum extensions in space and in momentum space belong to extremalous values of appropriate functions, are sets of *internal* states.

Thus, there is another

Point of generalization (2). Under which conditions not only the minimum, but *also* the maximum extensions in *all* states E are *internal* parameters and obtainable, like those, by means of the calculus? In such a case, according to the generalized

Helmholtz's rule and its complement, the total energy would become an internal system parameter, too.

3.4. Application of Euler's Principles of State Change to the Linear Harmonic Oscillator

3.4.1. The Principles of State Change for Classical Bodies

Following (Euler, II–5a), the changes of position and velocity during the time interval dt are

$$d\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} dt + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{M} \end{pmatrix}\begin{pmatrix} 0 \\ F \end{pmatrix} dt = \begin{pmatrix} v \\ \frac{1}{M}F \end{pmatrix} dt \tag{11}$$

The matrices of internal and external transformations,

$$\hat{U}_{\text{int}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \hat{U}_{\text{ext}} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{M} \end{pmatrix} \tag{12}$$

belong to the internal and external principles of state change, respectively. They do *not* commute: $\hat{U}_{\text{int}}\hat{U}_{\text{ext}} \neq \hat{U}_{\text{ext}}\hat{U}_{\text{int}}$. This means, that the internal and external transformations cannot be reduced onto each other, i.e., that the internal and the external principles of state change are *independent* of each other.

Equation (11) displays the following principles of state change of classical bodies (some of them were already described above). Up to order dt, it holds true, that

(CB1) the changes of state quantities (dv) depend only on the external causes (F) (mediated through the external transformation $\hat{U}_{\text{ext}}$), but not on nonstate quantities (x); in particular, $dv = 0$, if $F = 0$;

(CB2) the changes of the state quantities (dv) are independent of the state quantities (v) themselves;

(CB3) the changes of nonstate quantities (dx) depend directly only on state quantities (v) (mediated through the internal transformation $\hat{U}_{\text{int}}$); the external causes (F) affect the nonstate quantities (x) only indirectly (via state quantities, v);

(CB4) the changes of state (dv) and of nonstate quantities (dx) are independent of each other;

(CB5) as soon as the external causes vanish, the body remains in the state assumed in this moment: $Z(t) = \text{const} = Z(t_1) = v(t_1)$ for $t \geq t_1$, if $F(t) = 0$ for $t > t_1$.

As these principles reflect Eq. (11), we see here an example of the general rule, that the most fundamental differential equations w.r.t. time are of *first* order (cf. Enders, 1996).

Further, with $ddt = dF = 0$, the second-order changes read

$$dd\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} d\begin{pmatrix} x \\ v \end{pmatrix} dt$$
$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{M} \end{pmatrix}\begin{pmatrix} 0 \\ F \end{pmatrix} dt^2 = \begin{pmatrix} \frac{1}{M}F \\ 0 \end{pmatrix} dt^2 \qquad (13)$$

The upper line contains Newton's equation of motion ($M = \text{const}$); the lower line shows, that, up to second order in dt, an external force does not change the acceleration.

3.4.2. The Principles of State Change for Classical Conservative Systems

In this paragraph, we will show, that the principles of state conservation and state change for classical bodies, CB1 $\cdots$ CB5, apply *cum grano salis* to classical conservative systems as well.

Consider the linear harmonic oscillator as typical example. As state variable(s) we choose only the total energy, for

- taking the position, x, and the momentum, p, as state variables for a conservative system, one faces the problem, again, that the state is changed even in the absence of an external cause for doing so;
- taking the position, x, and the momentum, p, as state variables for statistical ensembles, one faces the problem of Gibb's paradox.

Correspondingly, the state function is the Hamilton function,

$$Z(t) = H(t, x(t), p(t)); \quad H(t, x(t), p(t)) = H_0(x(t), p(t)) + H_{\text{ext}}(t, x(t), p(t)) \qquad (14)$$

$H_0(x, p) = \frac{M}{2}\omega^2 x^2 + \frac{M}{2}p^2$. The "external causes" are described through $\partial H_{\text{ext}}/\partial t$ rather than H_{ext}, for if H_{ext} is time-independent, it should be absorbed into H_0; cf. Eq. (15) below.

Now, we will reformulate the principles CB1 $\cdots$ CB5 and show their validity through their compatibility with the Hamiltonian mechanics. Thus, up to order dt,

(CS1) the changes of state quantities (dH) depend only on the external causes ($\partial H_{\text{ext}}/\partial t$), but not on the changes of nonstate quantities (dx, dp); in particular, $dH = 0$, if $\partial H_{\text{ext}}/\partial t = 0$;

(CS2) the changes of the state quantities (dH) are independent of the state quantities (H) themselves;

(CS3) the changes of nonstate quantities (dx, dp) depend directly only on the state quantities (H); external causes ($\partial H_{\text{ext}}/\partial t$) affect the nonstate quantities (x, p) only indirectly (via the state quantities, H);

(CS4) the changes of the state (dH) and of the nonstate quantities (dx, dp) are independent of each other;

(CS5) as soon as the external causes ($\partial H_{\text{ext}}/\partial t$) vanish, the system remains in the state it has assumed in this moment: $H(t) = H(t_1) = \text{const}$ for $t \geq t_1$, if $\partial H_{\text{ext}}/\partial t = 0$ for $t \geq t_1$.

These principles imply the following *equation-of-state-change*.

$$dZ = dH = \frac{\partial H}{\partial p}\, dp + \frac{\partial H}{\partial x}\, dx + \frac{\partial H}{\partial t}\, dt \overset{!}{=} \frac{\partial H_{\text{ext}}}{\partial t}\, dt \tag{15}$$

Hence, $\frac{\partial H}{\partial p}\, dp + \frac{\partial H}{\partial x}\, dx = 0$ and, accounting for $\frac{dp}{dt} = -\frac{\partial V}{\partial x}$ in case of $H = T(p) + V(x, t)$, we have

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}; \quad \frac{dp}{dt} = -\frac{\partial H}{\partial x} \tag{16}$$

These are Hamilton's equations of motion.

Therefore, when reformulated in terms of appropriate state and nonstate variables, Euler's principles of state conservation and state change apply to conservative systems as well. Below, we will extend this capability to quantum systems.

3.4.3. Internal and External Parameters

In the standard solution of Newton's equation of motion for the harmonic oscillator,

$$x(t) = x_0 \cos(\omega t) + \frac{p_0}{M\omega} \sin(\omega t); \quad p(t) = p_0 \cos(\omega t) - M\omega x_0 \sin(\omega t) \tag{17}$$

the internal (M, ω) and external (x_0, p_0) parameters are interwoven. It should be enlightening to separate them.

This separation becomes straightforward, when we rewrite the solution in the form which would have been obtained from a solution of the Hamiltonian equations of motion as a *system* of equations (of first order in time).

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & \frac{1}{M\omega} \sin(\omega t) \\ -M\omega \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x_0 \\ p_0 \end{pmatrix} \overset{=}{_{def}} \hat{D}_\omega(t) \begin{pmatrix} x_0 \\ p_0 \end{pmatrix} \tag{18}$$

The matrix $\hat{D}_\omega(t)$ contains *only internal* parameters and the time, t. It describes rotations in phase space and exhibits the group property $\hat{D}_\omega(t) = \hat{D}_\omega(t - t') \cdot \hat{D}_\omega(t')$, $0 \leq t' \leq t$, which represents a point-mechanical analogue to Huygens' principle of wave propagation (cf. Enders, 1996).

We obtain an *un*coupled system of equations, if we go over to the variables

$$\tilde{x}(t) = \frac{1}{\sqrt{2}} \left[x(t) + \frac{i}{M\omega} p(t) \right]; \quad \tilde{p}(t) = \frac{1}{\sqrt{2}} \left[p(t) + iM\omega x(t) \right] \tag{19}$$

Their time-dependence is given through the eigenvalues of $\hat{D}_\omega(t)$.

$$\tilde{x}(t) = \tilde{x}(0)e^{i\omega t}; \qquad \tilde{p}(t) = \tilde{p}(0)e^{-i\omega t} \tag{20}$$

Obviously, the functions $\exp(\pm i\omega t)$ represent the *combined time-reversal symmetry* of space and momentum variables.

Moreover, one obtains immediately two independent first integrals of motion, viz.,

$$I_1 = e^{-i\omega t}\tilde{x}(t) = \tilde{x}(0); \qquad I_2 = e^{i\omega t}\tilde{p}(t) = \tilde{p}(0) \tag{21}$$

In fact, the energy equals $E = -i\omega I_1 I_2$, and, accordingly, the variables (19) *factorize* the Hamiltonian as

$$H(x, p) = \tilde{H}(\tilde{x}, \tilde{p}) = -i\omega\tilde{x}(t)\tilde{p}(t) = -i\omega\tilde{x}(0)\tilde{p}(0) \tag{22}$$

We will meet similar relationships within wave mechanics, again.

It is also noteworthy, that through the variables $\tilde{x}(t)$ and $\tilde{p}(t)$, the imaginary unit, $i = \sqrt{-1}$, appears not only as a mean to simplify calculations, but is indispensable for expressing physically relevant relationships.

4. QUANTIZATION AS SELECTION PROBLEM

Many derivations of QM from CM start from an *equation of motion* (which do *not* contain stationary-state variables) and, subsequently, face the problem of explaining the particularities of the *stationary* quantum states. As a consequence, various assumptions have to be made, which do not evolve out ouf CM itself. In this section, we will show, that such assumptions are not necessary, when one starts from the state description sketched in the aforementioned section and concentrates one the common rather than on the variant features of classical and quantum states. If one accepts, that energy conservation holds true for all isolated systems, then, the question arises, what is the energetic difference between classical and quantum systems and which are the selection criteria for a system to belong to this or to that group? It turns out, that quantization as selection problem addresses both the derivation and the solution of the stationary Schrödinger equation.

In order to replace the Bohr—Sommerfeld quantization conditions with a general quantization rule, Schrödinger has assumed, that

(A) a quantum-mechanical system be described by a wave function;
(B) this wave function obeys d'Alembert's wave equation with a nonclassical phase velocity depending on energy, E, and potential energy function, $V(x)$;
(C) Planck's relation between energy and frequency applies, i.e., $\nu = E/h$.

These assumptions are independent of each other and sufficient, but not necessary. For this, we will not use them, but instead the assumptions, that

(D1) Newton's axiomatics has to be replaced with Euler's axiomatics, in order to develop alternative equations of state change and alternative equations of motion;
(D2) energy conservation holds true not only for classical, but also for nonclassical conservative systems;
(D3) modifications of CM towards QM should not use concepts relying on the notion of path.

We will proceed according to the key prereqisite listed in the introduction. Since we will exploit a time-independent formulation of the energy conservation law, we will arrive at the *stationary* Schrödinger equation.

4.1. The Relationship Between CM and Non-CM as Selection Problem

Helmholtz's choice of the minus sign in Eq. (5) was motivated through the relation of the function $V(\mathbf{x})$ to the work, A. Therefore, it should be interesting to explore the *different physics*, which evolves out off the setting $E - T(\mathbf{p}) = -V(\mathbf{x})$ with $-\nabla V(\mathbf{x}) = \mathbf{F}(\mathbf{x})$ as before [switching to Euler's and Jacobi's use of $\tilde{V}(\mathbf{x}) = -V(\mathbf{x})$ with $+\nabla\tilde{V}(\mathbf{x}) = \mathbf{F}(\mathbf{x})$ involves only different signs and does not yet lead to different physics].

In fact, in place of Eq. (1), one can require

$$M\ddot{\mathbf{x}}(t) = -\mathbf{F}(\mathbf{x}(t)) \tag{23}$$

without coming into conflict with Newton's 1st and 3rd laws. Proceeding as after Eq. (1), one obtains $T(\mathbf{p}) = +V(\mathbf{x}) + \text{const}$ in place of $T(\mathbf{p}) = -V(\mathbf{x}) + \text{const}$ above. The result is a classical, but non-Newtonian mechanics, which is free of internal logical contradictions, though not realized in our world.

In what follows, we will represent these alternatives as *logical* relationships (cf. Enders and Suisky, in press). A unification of both alternatives will lead us to QM.

(1) *Helmholtzian condition of defining classical entities*
The difference $E - T(\mathbf{p})$ is semi-*definite*. In this case, the *exclusion*

$$\text{either} \quad E - T(\mathbf{p}) \geq 0 \quad \text{or} \quad E - T(\mathbf{p}) \leq 0 \tag{24}$$

holds true for *all* possible momentum configurations $\mathbf{p}$ in state E, i.e.,

$$\text{either} \quad \mathbf{p}|_{E-T(\mathbf{p})\geq 0} \in P^{\text{poss}} \quad \text{or} \quad \mathbf{p}|_{E-T(\mathbf{p})\leq 0} \in P^{\text{poss}} \tag{25}$$

In the domain $E - T(\mathbf{p}) \gtreqless 0$, we set $E - T(\mathbf{p}) = \pm V(\mathbf{x})$. Parametrizing $\mathbf{x} = \mathbf{x}(t)$, $\mathbf{p} = \mathbf{p}(t) = M\dot{\mathbf{x}}(t)$ and differentiating w.r.t. time, t, we obtain $\dot{\mathbf{p}}(t) = \mp\nabla V(\mathbf{x}(t)) = \pm\mathbf{F}(\mathbf{x}(t))$ corresponding to Eqs.(1) and (23), respectively. This means, that in the domain $\{\mathbf{p}|E - T(\mathbf{p}) \geq 0\}$, Newton's 2nd law applies, while in the domain, $\{\mathbf{p}|E - T(\mathbf{p}) \leq 0\}$, we deal with a classical, but non-Newtonian mechanics.

(2) *Helmholtzian condition of defining non-classical entities*
The difference $E - T(\mathbf{p})$ is *in*definite. Consequently, *both* inequations,

$$E - T(\mathbf{p}) \geq 0 \quad \text{and} \quad E - T(\mathbf{p}) \leq 0 \tag{26}$$

are *admitted* for the momentum configurations $\mathbf{p}$ in state E, i.e.,

$$\mathbf{p}|_{E-T(\mathbf{p})\geq 0} \in P^{\text{poss}} \quad \text{and} \quad \mathbf{p}|_{E-T(\mathbf{p})\leq 0} \in P^{\text{poss}} \tag{27}$$

The latter means $P^{\text{poss}} = P^{\text{all}}$.

According to our analysis of condition 1, P^{all} comprises domains with *different*, *incompatible* to each other equations of motion along paths $[\mathbf{x}(t), \mathbf{p}(t)]$. Consequently, systems with $P^{\text{poss}} = P^{\text{all}}$ do *not* move along paths.

(3) *The nonmechanical case*

$$\text{neither} \quad E - T(\mathbf{p}) \geq 0 \quad \text{nor} \quad E - T(\mathbf{p}) < 0 \tag{28}$$

for *any* momentum configuration $\mathbf{p}$ in state E. Here, no mechanics is possible, because $P^{\text{poss}} = \emptyset$.

Analogous or complementary statements can be made for the difference $E - V(\mathbf{x})$.

(1) *Complementary Helmholtzian condition of defining classical entities*
The difference $E - V(\mathbf{x})$ is semi-*definite*. In this case, the *exclusion*

$$\text{either} \quad E - V(\mathbf{x}) \geq 0 \quad \text{or} \quad E - V(\mathbf{x}) \leq 0 \tag{29}$$

holds true for *all* possible configurations $\mathbf{x}$ in state E, i.e.,

$$\text{either} \quad \mathbf{x}|_{E-V(\mathbf{x})\geq 0} \in C^{\text{poss}} \quad \text{or} \quad \mathbf{x}|_{E-V(\mathbf{x})\leq 0} \in C^{\text{poss}} \tag{30}$$

In the domain $E - V(\mathbf{x}) \overset{\geq}{\leq} 0$, we set $E - V(\mathbf{x}) = \pm T(\mathbf{p})$. Parametrizing $\mathbf{x} = \mathbf{x}(t)$, $\mathbf{p} = \mathbf{p}(t) = M\dot{\mathbf{x}}(t)$ and differentiating w.r.t. time, t, we obtain $\dot{\mathbf{p}}(t) = \mp\nabla V(\mathbf{x}(t)) = \pm\mathbf{F}(\mathbf{x}(t))$ like in Case 1. This means, that in the domain $\{\mathbf{x}|E - V(\mathbf{x}) \geq 0\}$, Newton's 2nd law applies, while in the domain, $\{\mathbf{x}|E - V(\mathbf{x}) \leq 0\}$, we deal with a classical, but non-Newtonian mechanics.

(2) *Complementary Helmholtzian condition of defining non-classical entities*
The difference $E - V(\mathbf{x})$ is *in*definite. Consequently, *both* inequations,

$$E - V(\mathbf{x}) \geq 0 \quad \text{and} \quad E - V(\mathbf{x}) \leq 0 \tag{31}$$

are *admitted* for the configurations $\mathbf{x}$ in state E, i.e.,

$$\mathbf{x}|_{E-V(\mathbf{x})\geq 0} \in C^{\text{poss}} \quad \text{and} \quad \mathbf{x}|_{E-V(\mathbf{x})\leq 0} \in C^{\text{poss}} \tag{32}$$

The latter means $C^{\text{poss}} = C^{\text{all}}$.

According to our analysis of condition 1, C^{all} comprises domains with *different*, *incompatible* to each other equations of motion along paths $[\mathbf{x}(t), \mathbf{p}(t)]$. Consequently, systems with $C^{\text{poss}} = C^{\text{all}}$ do *not* move along paths.

(3) *The complementary nonmechanical case*

$$\text{neither} \quad E - V(\mathbf{x}) \geq 0 \quad \text{nor} \quad E - V(\mathbf{x}) \leq 0 \tag{33}$$

for *any* configuration $\mathbf{x}$ in state E. Here, no mechanics is possible, because $C^{\text{poss}} = \emptyset$.

As a result, *alternative* definitions of the set of possible (momentum) configurations can be made, *without* coming into conflict with the classical axiomatics.

The six cases listed above correspond to the following hierarchy of *selection problems*.

1. A classical-mechanical system obeys *either* the laws of Newtonian CM, where $V(\mathbf{x}(t)) + T(\mathbf{p}(t)) = \text{const}$, $\dot{\mathbf{p}} = -\nabla V(\mathbf{x}) = +\mathbf{F}(\mathbf{x})$, *or* the laws of non-Newtonian CM, where $V(\mathbf{x}(t)) - T(\mathbf{p}(t)) = \text{const}$, $\dot{\mathbf{p}}(t) = +\nabla V(\mathbf{x}) = -\mathbf{F}(\mathbf{x})$ (Cases 1/1');
2. A mechanical system obeys *either* the laws of CM (motion along paths; Cases 1/1'), *or* the laws of non-CM (motion not along paths; Cases 2/2'); the logical opposition between the *xor* within CM and the *and* within non-CM has been discussed by Schrödinger (Schrödinger, 1933), it is a generalization of Bohr's and Heisenberg's complementarity between CM and QM to that between CM and non-CM;
3. A system is *either* a mechanical (Cases 1/1', 2/2') *or* a nonmechanical one (Cases 3/3').

Thus, the logical relationship (*xor*) between Newtonian CM and non-Newtonian CM is the same as that between Euler's axioms E1 and E2, as expressed in axiom E0. From this, we conclude, that these alternatives exclude each other *and*, at once, are "in harmony with each other," too. This harmony expresses itself in the fact, that the qualitative principles of state conservation and state change are the same (Euler's axioms, Newton's 1st and 3rd Laws), while the representations of the total energy and the equations of motion are different, though not without interrelations. These interrelations are provided through the use of the same dynamical variables $[\mathbf{x}(t)$ and $\mathbf{p}(t)]$ and the same functions of them (force, energies).

Similarly, the alternatives CM and non-CM exclude each other and, at once, are "in harmony with each other." The harmony is guaranteed through the energy law, while the representations of the total energy and the equations of motion should be different. We expect the occurrence of new entities, since, (i), new

dynamical variables are needed and, (ii), the possible alternatives of relationships between the energies are exhausted within CM. This will be explored next.

4.2. Derivation of the Stationary Schrödinger Equation as Selection Problem

4.2.1. Preliminaries

We have identified two closely related points, where a *natural and axiomatic* modification of CM towards QM (cf. Cases 2/2' above) is indicated.

Point of modification (1). Lifting the division of the set of (momentum) configurations into possible and impossible ones: $C^{\text{poss}} = C^{\text{all}}$ (Case 2'), $P^{\text{poss}} = P^{\text{all}}$ (Case 2).

Point of modification (2). Representing the extension and the total energy as *internal* parameters for *all* values of E, not only for $E = V_{\min}$ (state at rest).

According to our considerations above, there are two fundamental differences between classical and nonclassical mechanical systems. Comparing the differences between the nonclassical Cases 2/2' and the classical Cases 1/1', we note

Fundamental difference 1. The notion of *path* as a *point*-wise relationship between the coordinates in space and that in momentum space looses its meaning, for

- there is no longer a unique *algebraic* relation like $V(x) + T(p) = \text{const}$ between them;
- there is no longer a common parametrization like $x = x(t)$, $p = p(t) = M\dot{x}(t)$.

Furthermore, we observe

Fundamental difference 2. There are no longer coordinates $x_{\min\,/\,\max}$ ($p_{\min\,/\,\max}$) describing the *boundaries* of a system in (momentum) space, because the expressions $E - V(x)$ and $E - T(p)$ are no longer (semi-)definite [remember, that in Eqs. (7) and (8), the finite energy value limits the extension, and both the energy and the extensions in space and in momentum space are *external* parameters].

Both fundamental differences are interrelated as *both* imply, that

1. *all* (momentum) configurations should be considered *together*, and *all* configurations are related to *all* momentum configurations, and vice versa [for a system moving along a path, the path provides a *point-wise* relationship between the configurations and the momentum configurations, and the initial (momentum) configurations are *distinguished* from the other ones and can serve as *single* representatives for calculating the total energy of a *cl*assical system as $E_{\text{cl}} = V(x_0) + T(p_0)$];

2. *novel* entities are necessary for describing the extension of systems; presumably, these entities are *functions* of x and p, respectively, where the coordinates of their extremalous values play the role of $x_{\min\,/\,\max}$ ($p_{\min\,/\,\max}$).

This means, that we need another, a nonclassical representation of the total energy.

4.2.2. Energy of (Momentum) Configuration in the Nonclassical Case

Obviously, for a system for which all (momentum) configurations are possible, the function $V(x)$ $[T(p)]$ is no longer the contribution of the (momentum) configuration x (p) to the total energy, because $V(x)$ $[T(p)]$ is *un*bounded in the domain $C^{\text{poss}} = C^{\text{all}}$ $[P^{\text{poss}} = P^{\text{all}}]$. On the other hand, the energy is expected to be still composed of contributions of both x- and p-dependent terms (otherwise, the number of degrees of freedom would be lowered). These contributions be described through functions $V_{\text{ncl}}[C] = V_{\text{ncl}}(x)$ and $T_{\text{ncl}}[P] = T_{\text{ncl}}(p)$ to be specified now.

First, the functions $V_{\text{ncl}}(x)$ and $T_{\text{ncl}}(p)$ should interrelate energy and extension for a nonclassical system, although for a system the boundaries of which lie in the infinite, one can speak at most about an *effective* extension. This effective extension (we call it $x_{\max}$ and $p_{\max}$, respectively, again) should be connected with the maxima of the function $V_{\text{ncl}}(x)$ $[T_{\text{ncl}}(p)]$ as $V_{\text{ncl}}^{\max} = V_{\text{ncl}}(x_{\max})$ $[T_{\text{ncl}}^{\max} = T_{\text{ncl}}(p_{\max})]$. As in the classical case, any ordered set $x_{\max,1} < x_{\max,2} < \cdots$ with $V_{\text{ncl}}(x_{\max,1}) < V_{\text{ncl}}(x_{\max,2}) < \cdots$ or $p_{\max,1} < p_{\max,2} < \cdots$ with $T_{\text{ncl}}(p_{\max,1}) < T_{\text{ncl}}(p_{\max,2}) < \cdots$ should belong to the ordered set of states $E_1 < E_2 < \cdots$. From this follows, (i), that the function $V_{\text{ncl}}(x)$ $[T_{\text{ncl}}(p)]$ is *bounded* in the domain C^{poss} $[P^{\text{poss}}]$, and, (ii), that the values $x_{\max}$ $[p_{\max}]$ and, hence, the function $V_{\text{ncl}}(x)$ $[T_{\text{ncl}}(p)]$ depend (parametrically) on E. For this, we will write the nonclassical functions as $V_{E_{\text{ncl}}}(x)$ and $T_{E_{\text{ncl}}}(p)$.

The "harmony" between the classical and nonclassical functions is guaranteed, if the latter "contains" the former. For meeting the other requirements just listed, we introduce two dimensionless limiting multiplicative factors, $F_{E_{\text{ncl}}}(x)$ and $G_{E_{\text{ncl}}}(p)$, such that

$$V_{E_{\text{ncl}}}(x) = F_{E_{\text{ncl}}}(x) \cdot V(x) \leq E_{\text{ncl}}; \quad x \in C^{\text{all}} \tag{34a}$$

$$T_{E_{\text{ncl}}}(p) = G_{E_{\text{ncl}}}(p) \cdot T(p) \leq E_{\text{ncl}}; \quad p \in P^{\text{all}} \tag{34b}$$

This is the simplest possible modification of the classical expressions. The requirement $V_{\text{ncl}}(x)$, $T_{\text{ncl}}(p) \leq E_{\text{ncl}}$ results from the fact, that without paths, there is no compensation mechanism between x- and p-dependent terms.

The limiting factors $F_{E_{\text{ncl}}}(x)$ and $G_{E_{\text{ncl}}}(p)$ will be determined next.

4.2.3. The Nonclassical Representation of the Total Energy

Further, *all* (momentum) configurations enter the non-CM representation of the total energy and contribute to it, hence, the *nonclassical representation of the total energy* becomes

$$E_{\mathrm{ncl}} = \frac{\int_{C^{\mathrm{all}}} V_{E_{\mathrm{ncl}}}(x)\,dx}{\int_{C^{\mathrm{all}}} F_{E_{\mathrm{ncl}}}(x)\,dx} + \frac{\int_{P^{\mathrm{all}}} T_{E_{\mathrm{ncl}}}(p)\,dp}{\int_{P^{\mathrm{all}}} G_{E_{\mathrm{ncl}}}(p)\,dp}$$
$$= \frac{\int_{C^{\mathrm{all}}} F_{E_{\mathrm{ncl}}}(x)\cdot V(x)\,dx}{\int_{C^{\mathrm{all}}} F_{E_{\mathrm{ncl}}}(x)\,dx} + \frac{\int_{P^{\mathrm{all}}} G_{E_{\mathrm{ncl}}}(p)\cdot T(p)\,dp}{\int_{P^{\mathrm{all}}} G_{E_{\mathrm{ncl}}}(p)\,dp} \tag{35}$$

The denominators have been added for dimensional reasons. It is an *implicit*, therefore, *defining* equation for E_{ncl} [as well as for $F_{E_{\mathrm{ncl}}}(x)$ and $G_{E_{\mathrm{ncl}}}(p)$] containing *no external* parameters (remember, that within Bohr's theory, energy and extension of the stationary states are defined as *internal* parameters, too). The new functions $F_{E_{\mathrm{ncl}}}(x)$ and $G_{E_{\mathrm{ncl}}}(p)$ occur as *weights* for the contributions of the (momentum) configurations to the value of E_{ncl} (cf. Schrödinger, 1927, *Vierte Mitteilung*, §7).

The classical Eq. (6) is obtained formally from this representation, when setting $F_{E_{\mathrm{ncl}}}(x) \to F_{\mathrm{cl}}(x, x(t)) = \Omega_x \delta(x - x(t))$, $G_{E_{\mathrm{ncl}}}(p) \to G_{\mathrm{cl}}(p, p(t)) = \Omega_p \delta(p - p(t))$ (Ω_x, Ω_p normalization factors). This setting describes no limiting procedure, however. In particular, the functions F_{cl} and G_{cl} are not smooth and, thus, do not determine extensions.

In order to retain the "harmony" with CM mentioned above, in particular, the relation between energy and extension through ordered sets, the factors $F_{E_{\mathrm{ncl}}}(x)$ and $G_{E_{\mathrm{ncl}}}(p)$ should not change the signs of $V_{E_{\mathrm{ncl}}}(x)$ and $T_{E_{\mathrm{ncl}}}(p)$ against that of $V(x)$ and $T(p)$, i.e., they should be *nonnegative*. In order to free the subsequent calculations from this condition ('the less requirements, the better'), we set $F_{E_{\mathrm{ncl}}}(x) = \left|f_{E_{\mathrm{ncl}}}(x)\right|^2$ and $G_{E_{\mathrm{ncl}}}(p) = \left|g_{E_{\mathrm{ncl}}}(p)\right|^2$ (we will see later, that we have to deal with complex-valued functions). Inserting this into Eq. (35), we obtain the equivalent representation of the energy

$$E_{\mathrm{ncl}} = \frac{\int |f_{E_{\mathrm{ncl}}}(x)|^2 V(x)\,dx}{\int |f_{E_{\mathrm{ncl}}}(x)|^2 dx} + \frac{\int |g_{E_{\mathrm{ncl}}}(p)|^2 T(p)\,dp}{\int |g_{E_{\mathrm{ncl}}}(p)|^2 dp} \tag{36}$$

The new functions $f_{E_{\mathrm{ncl}}}(x)$ and $g_{E_{\mathrm{ncl}}}(p)$ occur as *weighting amplitudes* for the contributions of the (momentum) configurations to the value of E_{ncl}.

For all integrals in Eq. (36) to be finite, the functions $f_{E_{\mathrm{ncl}}}(x)$ and $g_{E_{\mathrm{ncl}}}(p)$ should obey the boundary conditions (from now on, we omit the index "ncl")

$$\lim_{p \to \pm\infty} g_E(p) = \lim_{x \to \pm\infty} f_E(x) = 0 \tag{37}$$

If $\lim_{|x| \to \infty} V(x) < \infty$, the boundary condition for $f_E(x)$ can be weakened.

4.2.4. The Stationary Schrödinger Equation

Now, since the functions $f_E(x)$ and $g_E(p)$ depend on the *same* variable parameter E, there is a certain relationship between them. One can show the Fourier transformation to be the only suitable relationship (Enders and Suisky, Einführung in die Quantenfeldtheorie des Festkörpers, manuscript in preparation).

$$f_E(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\frac{x\cdot p}{\Omega_x \Omega_p}} g_E(p) \frac{dp}{\Omega_p}; \quad g_E(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\frac{p\cdot x}{\Omega_x \Omega_p}} f_E(x) \frac{dx}{\Omega_x} \tag{38}$$

(Ω_x, Ω_p–normalization factors, again).

This enables us to eliminate the function $g_E(p)$ from Eq. (36) and to write ($\alpha \equiv \Omega_x \Omega_p$)

$$E = \frac{\int \bar{f}_E(x) \hat{H}(x) f_E(x)\, dx}{\int |f_E(x)|^2\, dx}; \quad \hat{H}(x) \equiv H\left(x,\, p \to -i\alpha \frac{\partial}{\partial x}\right) \tag{39}$$

or

$$\int_{-\infty}^{+\infty} \bar{f}_E(x) \left[V(x) f_E(x) - \frac{\alpha^2}{2M} \frac{\partial^2}{\partial x^2} f_E(x) - E f_E(x) \right] dx = 0 \tag{40}$$

A sufficient condition for this equation to hold true is the vanishing of the expression within the square brackets (in case of $\bar{f}_E(x)$ being linearly independent of $f_E(x)$, this condition is even necessary).

$$V(x) f_E(x) - \frac{\alpha^2}{2M} \frac{\partial^2}{\partial x^2} f_E(x) - E f_E(x) = 0 \tag{41}$$

To take this equation as a condition for the validity of the nonclassical representations of the energy law given above is supported by the well-known fact, that this equation holds true for the minimum value of the r.h.s. of Eq. (39), i.e., for the ground state.

In turn, when Eq. (41) holds true, we have

$$\lim_{R \to +\infty} \frac{\int_{-R}^{+R} \bar{f}_E(x) \hat{H}(x) f_E(x)\, dx}{\int_{-R}^{+R} |f_E(x)|^2\, dx} = E \tag{42}$$

independent of the boundary conditions for $f_E(x)$. The only remaining conditions are two-fold differentiability and existence of a Fourier transform $g_E(p)$. Indeed, the state equation should hold true independent of boundary conditions when

claiming applicability to all systems and states, respectively, both bounded and unbounded.

Identifying $\alpha = \hbar$ and $f_E(x) = \sqrt{\Omega_x}\psi_E(x)$, Eq. (41) becomes the *stationary Schrödinger equation*. This identification is justified by comparing its solution for the harmonic oscillator with Planck's oscillator (Enders and Suisky, Einführung in die Quantenfeldtheorie des Festkörpers, manuscript in preparation). The momentum representation of the stationary Schrödinger equation is obtained analogously. Note, that this derivation also justifies the use of the *classical* expressions $V(x)$ and $T(p)$, among this the identification of Planck's frequency with the classical oscillator frequency, $\omega_{\mathrm{cl}} = \sqrt{\kappa/M}$.

4.3. Solution of the Stationary Schrödinger Equation as Selection Problem

On principle, accounting for the boundary conditions (37), Eq. (41) can be solved as a classical eigenvalue problem (Schrödinger, 1927). However, this would discard the nonclassical content of the wave function. For this, we look for a solution method, which disclosures the discreteness *without* using boundary conditions like (37). As in the CM selection problems, there is a mathematical part to be solved first and a physical part yielding the final solution.

4.3.1. The Mathematically Distinguished Solutions

Introducing dimensionless variables, one obtains for the linear harmonic oscillator Weber's equation being one of the equations of the parabolic cylinder (Abramowitz and Stegun, 1964; Whittaker and Watson, 1927).

$$\frac{d^2u_\nu(\xi)}{d\xi^2} + \left(\nu + \frac{1}{2} - \frac{1}{4}\xi^2\right)u_\nu(\xi) = 0; \qquad \nu \equiv \frac{E}{\hbar\omega} - \frac{1}{2} \tag{43}$$

Here, the "energy parameter" is primarily a mathematical parameter of the differential equation considered, it becomes a physical parameter only after passing suitable criteria (cf. Schrödinger, 1927).

Now, for and only for the values $\nu = -1$ and $\nu = 0$ the l.h.s. of Eq. (43) *factorizes*.

$$\begin{gathered}\left(\frac{d}{d\xi} + \frac{1}{2}\xi\right)\left(-\frac{d}{d\xi} + \frac{1}{2}\xi\right)u_{-1}(\xi) = 0; \\ \left(-\frac{d}{d\xi} + \frac{1}{2}\xi\right)\left(\frac{d}{d\xi} + \frac{1}{2}\xi\right)u_0(\xi) = 0\end{gathered} \tag{44}$$

Note, that these factors are closely related to the classical variables (19)]. Therefore, the values $\nu = -1$ and $\nu = 0$ are *mathematically distinguished* from all other ν-values. The corresponding solutions, $u_{-1}(\xi)$ and $u_0(\xi)$, are mathematically

equivalent, but physically different. $u_0(\xi) = u_0(0)\exp\left(-\frac{1}{4}\xi^2\right)$ is a limiting function, while $u_{-1}(\xi) = u_{-1}(0)\exp\left(+\frac{1}{4}\xi^2\right)$ is not. This distinguishes physically the value $\nu = 0$ over the value $\nu = -1$.

If there would be no other distinguished ν-values, there would be only *one* state ($\nu = 0$). However, a system having got just *one* state could not exchange energy with its environment. Hence, there should be further distinguished ν-values. In order to find them, we examine the *recurrence relations* for the *e*ven and *o*dd solutions to Eq. (43) (cf. Abramowitz and Stegun 1964, §19.6),

$$\left(\frac{d}{d\xi} + \frac{\xi}{2}\right) u_\nu^e(\xi) = -\nu u_{\nu-1}^o(\xi); \quad \left(-\frac{d}{d\xi} + \frac{\xi}{2}\right) u_{\nu-1}^o(\xi) = -u_\nu^e(\xi) \tag{45}$$

and

$$\left(-\frac{d}{d\xi} + \frac{\xi}{2}\right) u_\nu^e(\xi) = (\nu+1)\, u_{\nu+1}^o(\xi); \quad \left(\frac{d}{d\xi} + \frac{\xi}{2}\right) u_{\nu+1}^o(\xi) = u_\nu^e(\xi) \tag{46}$$

These recurrence relations base on Whittaker's representation of the solutions as contour integrals (Whittaker and Watson, 1927)—a method being developed well before QM, i.e., independently of the needs of QM and, thus, for the *whole* interval $-\infty < \nu < +\infty$. They

- do *not* follow from the usual solution methods;
- interrelate solution functions with *finite* difference between their ν-value, viz., $\Delta\nu = \pm 1$ ($\Delta E = \pm\hbar\omega$);
- divide the continuum of ν-values into over-countably infinitely many sets of countably infinitely many ν-values each:
  Set (a): $\{\ldots, -3, -2, -1\}$ [the relations (46) breaks at $\nu = -1$ (being one of the two distinguished values found above)];
  Set (b): $\{0, +1, +2, \ldots\}$ [the relations (45) breaks at $\nu = 0$ (being the other distinguished value found above)];
  Sets (c): $\{\ldots, -2 + \nu_r, -1 + \nu_r, \nu_r, \nu_r + 1, \nu_r + 2, \ldots \mid -1 < \nu_r < 0\}$ [ν_r - reference value; *no* break].
- reflect the *genuine discrete structure* saught for (they have nothing to do with boundary conditions, since *all* solutions exhibit this structure, not only Schrödinger's eigensolutions).

The smallest interval representing *all* solutions is the closed interval $\nu_r = [-1, 0]$. Here, again, the values $\nu_r = -1$ (cf. set a) and $\nu_r = 0$ (cf. set b) are mathematically distinguished, this time, as being the boundary points of this interval. All inner interval points, $-1 < \nu_r < 0$ (cf. sets c), are mathematically equivalent among each other and, consequently, *not* distinguished mathematically.

Hence, the sets (a) and (b) are mathematically distinguished, when compared with sets (c).

4.3.2. The Physically Distinguished Solutions

The two mathematically distinguished sets (a), $\nu = -1, -2, \ldots$, and (b), $\nu = 0, +1, +2, \ldots$, are mathematically equivalent. Physically, however, they are *different*. All functions of set (b) are limiting functions, while all functions of set (a) are not [those of set (c) are also not]. An even stronger selection criterion is provided by the energy law.

As a matter of fact, the transition of the oscillator from state $E_\nu = \left(\nu + \frac{1}{2}\right)\hbar\omega$ to state $E_{\nu-1} = \left(\nu - \frac{1}{2}\right)\hbar\omega$ is connected with the delivery of the energy $\hbar\omega$ to the environment, e.g., to a radiation field. Hence, an oscillator with the possible energy values $E_{-1} = -\frac{1}{2}\hbar\omega$, $E_{-2} = -\frac{3}{2}\hbar\omega, \cdots$ [set (a)] could deliver an *infinite* amount of energy to the environment, in contradiction to the energy law (Helmholtz, 1847). In other words, these transitions must stop at some *minimum* amount of energy content of the oscillator. Such a minimum ["permanent/ground state" (Bohr, 1913), "normal state" (Heisenberg, 1925)] is contained soleily in set (b); $E_0 = \frac{1}{2}\hbar\omega$, $E_1 = \frac{3}{2}\hbar\omega, \cdots$. Hence, set (b) is *physically distinguished* from set (a) through its *compatibility with the energy law*.

This means, that the physical solutions are that with $\nu = n = 0, 1, 2, \ldots$. This agrees with Schrödinger's result, of course, but it has been obtained through analyzing the stationary Schrödinger equation *alone*, without resorting to boundary conditions (and without computing the actual solution functions).

The corresponding functions $u_n(\xi)$ can be calculated most simply from Eq. (45). One obtains the well-known result $u_n(\xi) = \exp\left\{-\frac{1}{4}\xi^2\right\} He_n(\xi)$ (He_n—nth Hermite polynomial), where Schrödinger's boundary conditions, $u_n(\xi) \to 0$ for $\xi \to \pm\infty$, are fulfilled automatically.

The values, $x_{\max,n}$, where the functions $\psi_n^2(x)V(x)$ assume their maximum, increase with n: $x_{\max,1} < x_{\max,2} < \cdots$. This means, that the ordering relation between energy and effective extension in space and in momentum space is realized as described in the generalized Helmholtz's rule; the minimum extension is assumed in the ground state.

Finally, using the recurrence formula $He_{n+1}(x) = x \cdot He_n(x) - n \cdot He_{n-1}(x)$ and the inequality $|He_n(x)| < e^{x^2/4}\sqrt{n!}k$ ($k \approx 1.086435$) (cf. Abramowitz and Stegun, 1964, 22.14.17), one can prove, that $V_{\text{ncl}}(x)$, $T_{\text{ncl}}(p) \leq E_{\text{ncl}}$, as required above. Actually, $u_n^2(\xi) \cdot \frac{1}{4}\xi^2 < n + \frac{1}{2}$ ($-\infty < \xi < +\infty$, $n = 0, 1, \ldots$). The occurrence of the "smaller than" sign means, that—in contrast to the classical oscillator—the energy cannot be represented by one single (momentum) configuration; therefore, the quantum oscillators does *not* exhibit a state at rest.

Note, that these results follow alone from the most general principles of state description according to Leibniz, Euler, Helmholtz and Schrödinger, without solving any state equation or equation of motion and without assuming particular boundary conditions.

5. THE TIME-DEPENDENT CASE

While Schrödinger (1927) and Heisenberg (1925) started from a time-dependent equation, we have worked so far with the set of all possible (momentum) configurations of systems in their stationary states, where time plays no role. In order to incorporate the time, we will proceed as in the classical case and consider first the stationary states, then, the change of change and, finally, arrive at the time-dependent Schrödinger equation as the equation of motion.

5.1. Time-Dependence of the Stationary States

The only place for implementing a time-dependence into the stationary quantum state dealt with so far is the wave function, for which we thus write $f_E(x,t)$ and $g_E(p,t)$, respectively.

The invariance of the energy (36) implies, that the time-dependence of the wave functions is of separative multiplicative form

$$f_E(x,t) = f_E(x) \cdot \theta_E(t); \quad g_E(p,t) = g_E(p) \cdot \theta_E(t); \quad |\theta_E(t)| = 1 \tag{47}$$

The factor $\theta_E(t)$ is the same for *both* functions, $f_E(x,t)$ and $g_E(p,t)$, since these are for the same reasons interrelated through the Fourier transformation, for which the functions $f_E(x)$ and $g_E(p)$ are. $|\theta_E(t)| = 1$, because $|\theta_E(t)| \neq \text{const}$ would imply, that the (still stationary) functions $F_E(x,t)$ and $G_E(p,t)$ are time-dependent. Moreover, a real-valued function $\theta_E(t)$ would depend on the normalization, $f_E(x,t) \to f_E(x,t)/\|f_E(x,t)\|^{1/2}$ [the r.h.s. of Eq. (36) is invariant against $f_E(x,t) \to \|f_E(x,t)\|^{\text{number}} \cdot f_E(x,t)$]. In other words, the form $\theta_E(t) = \exp\{i\chi_E(t/t_E)\}$ is necessary for the existence of stationary states. Here, χ_E is the phase to be determined next. Since it is dimensionless, there is a reference time, t_E, which we expect to depend on energy, E.

Since this time dependence is not related to a new quantization problem, there is no reason for a new quantum constant, say, h', so that the reference time should equal $t_E = \text{const} \cdot h/E$. And because $\theta_E(t)$ describes the time-dependence for both the spatial and the momentum functions, it should obey the combined time-reversal symmetry found in Eq. (20), i.e., $\theta_E(-t) = \theta_E(t)^*$. Hence, $\chi_E(-t/t_E) = -\chi_E(t/t_E)$. For free particles, we have $g_E(p,t) \sim \delta(p - \hbar k)e^{-i\omega t}$ and $f_E(x,t) \sim e^{i(kx-\omega t)}$, where the group velocity equals the time-*in*dependent particle velocity, $v_g = d\omega/dk = \hbar k/m$ (m—mass). This implies $\chi_E(t/t_E) = Et/\hbar$, $\theta_E(t) = \exp\{-iEt/\hbar\}$ and, finally,

$$f_E(x,t) = f_E(x)e^{-i\frac{Et}{\hbar}}; \quad g_E(p,t) = g_E(p)e^{-i\frac{Et}{\hbar}} \tag{48}$$

As a consequence, the (still stationary) functions $f_E(x,t)$ and $g_E(p,t)$ obey the time-dependent Schrödinger equation in location and momentum

representation, respectively; we thus identify $f_E(x,t) = \sqrt{\Omega_x}\psi_E(x,t)$ and $g_E(p,t) = \sqrt{\Omega_p}\phi_E(p,t)$.

For later use, we write down the operator form of this time dependence as

$$\psi_E(x,t) = \hat{U}_0(x;t)\psi_E(x,0), \quad \text{where} \quad \hat{U}_0(x;t) = e^{-i\hat{H}_0(x)t/\hbar} \tag{49}$$

is the (unitary) time-development operator, and $\hat{H}_0\psi_E(x,t) = E\psi_E(x,t)$.

However, at this stage, these are only kinematic, not dynamic relations; hence, it is not justified to simply carry over them to the nonstationary case.

5.2. The Equation-of-State-Change

Thus, in order to *derive* rather than to postulate dynamic laws, we exploit Euler's principles of state change (the principle of least action is not appropriate here, because, within QM, the minimum action is governed by the value of h).

Since the Hamilton function is a suitable classical state function, it is certainly reasonable to exploit the nonclassical representation (36) as nonclassical state function. The 'external causes' be given through $\partial V_{\text{ext}}(x,t)/\partial t$ [as in the classical case, if V_{ext} is time-independent, it should be absorbed into $V(x)$]. Then, the nonclassical state function reads

$$\begin{aligned} Z_{\text{ncl}}(t) &= \frac{\langle\psi(x,t)|V(x) + V_{\text{ext}}(x,t)|\psi(x,t)\rangle}{\langle\psi(x,t)|\psi(x,t)\rangle} + \frac{\langle\phi(p,t)|T(p)|\phi(p,t)\rangle}{\langle\phi(p,t)|\phi(p,t)\rangle} \\ &= \frac{\langle\psi(x,t)|\hat{H}(x,t)|\psi(x,t)\rangle}{\langle\psi(x,t)|\psi(x,t)\rangle}; \quad \hat{H}(t,x) = \hat{H}_0(x) + V_{\text{ext}}(t,x) \end{aligned} \tag{50}$$

Here, $\psi(x,t)$ is the general time-dependent wave function; the energy, E, is no longer a characteristic parameter of the system and thus not indicated. For continuity reasons (see principle QS5 below), it is the Fourier transform of the function $\phi(p,t)$. In the stationary case, we have $\psi(x,t) = \psi_E(x,t)$, $Z_{\text{ncl}}(t) = E = \text{const}$ and $dZ = 0$.

Now, we reformulate the Eulerian principles of state change for classical systems, CS1$\cdots$CS5, in terms of the *non*-classical state (Z—we will omit the index *ncl*) and nonstate functions (ψ, $\bar{\psi}$). This means, that, up to order dt,

- **(QS1)** the changes of state quantities (dZ) depend only on external ($\partial V_{\text{ext}}/\partial t$), but not on internal causes ($\hat{H}_0$); in particular, $dZ = 0$, if $\partial V_{\text{ext}}/\partial t = 0$;
- **(QS2)** the changes of the state quantities (dZ) are independent of the state quantities (Z) themselves;
- **(QS3)** the changes of nonstate quantities ($d\psi$, $d\bar{\psi}$) depend directly only on state quantities (Z); the external causes ($\partial V_{\text{ext}}/\partial t$) affect nonstate quantities (ψ, $\bar{\psi}$) only indirectly (via the changes of state quantities, dZ);
- **(QS4)** the changes of the state quantities (dZ) and of the nonstate quantities ($d\psi$, $d\bar{\psi}$) are indepent of each other;

(QS5) as soon as the external causes $(\partial V_{\text{ext}}/\partial t)$ vanish, the system remains in the state it has assumed in this moment; in case of that being an eigenstate, say, E_1, this means $Z(t) = \text{const} = Z(t_1) = E_1$ and, as a consequence, $\psi(x,t) = \psi_{E_1}(x,t) = \psi_{E_1}(x,t_1)\exp\left\{-\frac{i}{\hbar}E_1\,(t-t_1)\right\}$ for $t \geq t_1$, if $\partial V_{\text{ext}}(x,t)/\partial t = 0$ for $t \geq t_1$.

Then, the *equation-of-state-change* becomes

$$dZ = \frac{\langle d\psi|\hat{H}|\psi\rangle + \langle\psi|d\hat{H}|\psi\rangle + \langle\psi|\hat{H}|d\psi\rangle}{\langle\psi|\psi\rangle} - \frac{\langle d\psi|\psi\rangle + \langle\psi|d\psi\rangle}{\langle\psi|\psi\rangle^2}\langle\psi|\hat{H}|\psi\rangle$$
$$\stackrel{!}{=} \frac{\langle\psi|d\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\langle\psi|\frac{\partial}{\partial t}\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle}\,dt \tag{51}$$

5.3. The Equation of Motion

The equation-of-state-changesatisfies the principles QS1$\cdots$QS5, i.e., $dZ = \langle\psi|\psi\rangle^{-1}\,\langle\psi|\frac{\partial}{\partial t}\hat{H}|\psi\rangle dt$, if

$$\langle d\psi|\hat{H}|\psi\rangle + \langle\psi|\hat{H}|d\psi\rangle = 0 \quad \text{and} \quad \langle d\psi|\psi\rangle + \langle\psi|d\psi\rangle = 0 \tag{52}$$

In order to satisfy these equations, we set, accounting for Eq. (49) and QS5,

$$\begin{aligned} \psi(x,t) &= \hat{U}(x;t)\psi(x,0); \quad \hat{U}(x;0) = 1; \\ d\psi(x,t) &= d\hat{U}(x;t)\psi(x,0) = \frac{\partial}{\partial t}\hat{U}(x;t)\psi(x,0)\,dt \end{aligned} \tag{53}$$

As in the stationary case, the homogeneity in ψ corresponds to the fact, that only the relative values of $\psi(x,t)$ exhibit a physical meaning, see Eq. (50).

Inserting these expressions into Eqs.(52) yields

$$\begin{aligned} &\langle d\hat{U}(t)\psi(x,0)|\hat{H}|\hat{U}(t)\psi(x,0)\rangle + \langle\hat{U}(t)\psi(x,0)|\hat{H}|d\hat{U}(t)\psi(x,0)\rangle = 0 \\ &\text{and} \quad \langle d\hat{U}(t)\psi(x,0)|\hat{U}(t)\psi(x,0)\rangle + \langle\hat{U}(t)\psi(x,0)|d\hat{U}(t)\psi(x,0)\rangle = 0 \end{aligned} \tag{54}$$

Obviously, $d\hat{U} = i\gamma f(\hat{H})\,dt$ (γ—real-valued t-independent constant, f—entire-rational function) is a solution to both equations. Compatibility with the stationary case (49) implies $\gamma = -1/\hbar$ and $f(\hat{H}) = \hat{H}$. Hence,

$$d\hat{U}(x;t) = \frac{-i}{\hbar}\hat{H}(x;t)\,dt; \quad \hat{U}(x;t) = P\left(\exp\left\{\frac{-i}{\hbar}\int_0^t \hat{H}(x,t')dt'\right\}\right) \tag{55}$$

where P denotes Dyson's time-ordering operator (Dyson, 1949). Using this solution, formula (53) yields immediately the time-dependent Schrödinger equation for $\psi(x,t)$. Analogously, its momentum representation can be derived.

Both representations of the time-dependent Schrödinger form the *equation of motion*. As in the classical case, the equation of motion is a dynamic equation for *non*state entities.

5.4. The Invariant Expression in the Nonstationary Case

Alternatively, one can derive the nonstationary Schrödinger equation from an invariant expression, that replace the nonclassical representation of the energy (but reduces to that representation in the stationary case).

Like above, the 'external causes' be described by means of time-dependent functions $V_{\text{ext}}(x, t)$ and, moreover, $T_{\text{ext}}(p, t)$. Looking, again, for an invariant functional of the limiting functions, we make the ansatz

$$\frac{\int \bar{f}(x,t)V(x,t)f(x,t)\,dx}{\int \bar{f}(x,t)\frac{\partial}{\partial t}f(x,t)\,dx} + \frac{\int \bar{g}(p,t)T(p,t)g(p,t)\,dx}{\int \bar{g}(p,t)\frac{\partial}{\partial t}g(p,t)\,dx} = \beta \stackrel{!}{=} \text{const} \quad (56)$$

The time-derivative is necessary for avoiding an instantaneous reaction of the system. It is of 1st order, because the stationary case yields only the initial values $f(x, 0)$ and $g(p, 0)$. The compatibility with the stationary case (48) yields $\beta = i\hbar$. Proceeding as in the stationary case, we obtain the pair of the location and of the momentum representations of the time-dependent Schrödinger equation.

We remark, that the identification $\beta = i\hbar$ can also be derived from the time-dependent equation invoking special-relativistic arguments (here, the speed of light provides the relationship between spatial and temporal variables).

6. SYSTEMS OF TWO EQUAL SUBSYSTEMS

We will sketch, how the Euler–Helmholtzian manner of (stationary) state description adopted in this paper, in particular, the nonclassical representation of energy (36) can be exploited for the description of quantum many-body systems. *No* additional assumptions will be made, but the rather natural one, that—as in single-particle systems—the symmetries of the classical system exist also in the corresponding nonclassical system.

When expressed in generalized coordinates and momenta, the potential and kinetic energy functions of a conservative system with two equal classical bodies exhibit the symmetry $V(q_2, q_1) = V(q_1, q_2)$ and $T(p_2, p_1) = T(p_1, p_2)$, respectively, and the total energy—as well as the other integrals of motion—of this system is invariant against an exchange of (the labels of) the two bodies (thus, the Lagrangean formalism is a prerequisite of many-body quantum mechanics). It is natural to assume, that this permutation symmetry of the stationary and nonstationary states applies to the corresponding nonclassical system as well.

As a consequence, we have for the energy of the nonclassical two-body system (returning to the variable x) the representation

$$E = \frac{\int F_E(x_1, x_2)V(x_1, x_2)\,dx_1dx_2}{\int F_E(x_1, x_2)\,dx_1dx_2} + \frac{\int G_E(p_1, p_2)T(p_1, p_2)\,dp_1dp_2}{\int G_E(p_1, p_2)\,dp_1dp_2}$$
$$= \frac{\int F_E^+(x_1, x_2)V(x_1, x_2)\,dx_1dx_2}{\int F_E^+(x_1, x_2)\,dx_1dx_2} + \frac{\int G_E^+(p_1, p_2)T(p_1, p_2)\,dp_1dp_2}{\int G_E^+(p_1, p_2)\,dp_1dp_2} \quad (57)$$

Here, $F_E^+(x_1, x_2) = \frac{1}{2}[F_E(x_1, x_2) + F_E(x_2, x_1)]$ and $G_E^+(p_1, p_2) = \frac{1}{2}[G_E(p_1, p_2) + G_E(p_1, p_2)]$ are the symmetrical parts of the weight functions $F_E(x_1, x_2)$ and $G_E(p_1, p_2)$. Their antisymmetrical parts, $F_E^-(x_1, x_2) = \frac{1}{2}[F_E(x_1, x_2) - F_E(x_2, x_1)]$ and $G_E^-(p_1, p_2) = \frac{1}{2}[G_E(p_1, p_2) - G_E(p_1, p_2)]$, do not contribute to the integrals (since all integrands are continuous and bounded, the order of integration can be interchanged).

Tentatively, one expects $F_E^-(x_1, x_2) \equiv 0$ and $G_E^-(p_1, p_2) \equiv 0$. In fact, let us consider the weighted distance,

$$\langle x_1 - x_2\rangle = \frac{\int F_E(x_1, x_2)(x_1 - x_2)\,dx_1dx_2}{\int F_E(x_1, x_2)\,dx_1dx_2} = \frac{\int F_E^-(x_1, x_2)(x_1 - x_2)\,dx_1dx_2}{\int F_E^+(x_1, x_2)\,dx_1dx_2} \quad (58)$$

Since both bodies/particles are equal, there is no reason for the case $\langle x_1 - x_2\rangle > 0$ to be preferred over the case $\langle x_1 - x_2\rangle < 0$. For this, $\langle x_1 - x_2\rangle = 0$ and, consequently, $F_E^-(x_1, x_2) \equiv 0$.

An analogous consideration of the entity $\langle p_1 - p_2\rangle$ leads to the result $G_E^-(p_1, p_2) \equiv 0$.

In other words, the weight functions are *symmetric*. For the weight amplitudes, this means

$$|\psi_E(x_2, x_1)|^2 = |\psi_E(x_1, x_2)|^2; \quad |\phi_E(p_2, p_1)|^2 = |\phi_E(p_1, p_2)|^2 \quad (59)$$

Since this symmetry applies to *all* systems under consideration, it implies, that the weight amplitudes themselves are *either symmetric*,

$$\psi_E(x_1, x_2) = \psi_E^+(x_1, x_2) \equiv \tilde{\psi}_E(x_1, x_2) + \tilde{\psi}_E(x_2, x_1);$$
$$\phi_E(p_1, p_2) = \phi_E^+(p_1, p_2) \equiv \tilde{\phi}_E(p_1, p_2) + \tilde{\phi}_E(p_2, p_1) \quad (60)$$

or anti-symmetric,

$$\psi_E(x_1, x_2) = \psi_E^-(x_1, x_2) \equiv \tilde{\psi}_E(x_1, x_2) - \tilde{\psi}_E(x_2, x_1);$$
$$\phi_E(p_1, p_2) = \phi_E^-(p_1, p_2) \equiv \tilde{\phi}_E(p_1, p_2) - \tilde{\phi}_E(p_2, p_1) \quad (61)$$

where, the tilded functions are *free* of the requirement of permutation symmetry.

As a consequence, there are two classes of wave functions (cf. Pauli, 1973), $\psi_E^+(x_1, x_2) = \mathfrak{F}[\phi_E^+(p_1, p_2)]$ and $\psi_E^-(x_1, x_2) = \mathfrak{F}[\phi_E^-(p_1, p_2)]$, respectively ($\mathfrak{F}[\cdot]$

means the 2D Fourier transformation). The properties of them are well known and need not be discussed here.

The implications of this approach for the issue of (in)distinguishability of classical and quantum particles will be discussed elsewhere.

7. SUMMARY AND CONCLUSIONS

We have presented a novel representation of quantization as a selection rather than eigenvalue problem. It starts from Euler's rather than from Newton's axiomatics, uses Helmholtz's treatment of the energy conservation law, generalizes Euler's method of maxima and minima, and realizes Einstein's criterion for the difference between the energy spectra of classical and quantum systems.

Mechanical systems can be divided into the two classes of classical and nonclassical systems. The internal system parameters like mass and potential constant are the same in both classes. And in both classes, the total energy is a state variable ("state" in the sense of Newton and Euler, i.e., stationary state). The fundamental *difference* between both classes is the different representation of the total energy in terms of internal (including universal) and external parameters and the manner of state change. The classical state equation, $H(p(t), x(t)) = E$ defines the energy, E, as explicit, continuous function of the external, continuous parameters $x_0 = x(t_0)$ and $p_0 = p(t_0)$ (the initial conditions); consequently, it is also an external, continuous parameter. In contrast, the nonclassical state equation, $E = < \psi_E(x)|\hat{H}(x)|\psi_E(x) > / < \psi_E(x)|\psi_E(x) >$, defines the energy implicitly as it contains no external parameters. As a consequence, the energy of quantum systems is an internal and essentially discontinuous parameter.

These different representations of the energy can be obtained only through different additional assumptions on the systems described. It is often assumed, that the difference is caused by the existence of the action parameter. We have shown, that this assumption is not necessary. It is sufficient to make different assumptions about the (momentum) configurations a mechanical system may assume and their relations to the total energy.

Within CM, the physical spectrum of the mathematical "energy parameter" is a continuum; for the oscillator, $-\frac{1}{2} \leq \nu < \infty$. The quantization procedure proposed in this paper shows, that within QM, this continuum is excluded *even mathematically* from the physical spectrum. As a consequence, the noninteger values of the energy parameter are even not available for the physical selection of the spectrum. In other words, the physical selection does not deal with the continuum, so that a question like "what about the physically discarded continuous energy values?" does not arise at all. The physically discarded values $\nu = -1, -2, \ldots$ are not part of the classical spectrum ($\nu \geq -\frac{1}{2}$).

Because of the methodological advantages of this approach, three important methodical problems stressed by Schrödinger are solved.

1. The "quantum equation" should "carry the quantum conditions in itself" (cf. Schrödinger, 1927, Second Contribution, p. 511), i.e., strictly speaking, independent of the boundary conditions [note, that Schrödinger's quantization conditions are—like de Broglie's ones, which interprete Bohr's "quantum orbits" as standing waves (de Broglie, 1925)—to a large amount classical boundary conditions, cf. (Heisenberg, 1977)];
2. There should be a special mathematical method for solving the stationary Schrödinger equation, which accounts for the nonclassical character of the quantization problem, i.e., which is different from the classical methods for calculating the eigenmodes of strings, resonators and so on, cf. (Schrödinger, 1927, Second Contribution, p. 513). The determination of the solution without using the boundary conditions shows, that they exhibit "maximum strength" in the sense of (Einstein, 1977);
3. The derivation should uniquely decide, that the energy rather than the frequency values are discretized. For if one replaces E with $h\nu$ instead of $h\nu$ with E, one obtains the stationary equation $H\psi = h\nu\psi$, whence, *primarily*, the frequency, ν, would be discretized; frequency discretization, however, is a *classical* phenomenon cf. (Schrödinger, 1927, Second Contribution, pp. 511, 519).

On discussing the pathes for classical and quantum particles, Schrödinger concluded, that

> We are faced here with the full force of the logical opposition between an
>
> *either–or* (point mechanics)
>
> and a
>
> *both–and* (wave mechanics)
>
> This would not matter much, if the old system were to be dropped entirely and to be replaced by the new. Unfortunately, this is not the case. (Schrödinger, 1933)

A formulation of this "*both–and*" can be seen in the path integral representation (Feynman, 1949), cf. (Lübbig, 1999). The exploitation of configurations rather than paths exhibits the advantage, that the former ones apply in both CM and QM, while the latter ones do not. More important, however, Schrödinger's consideration means, that only a deeper analysis of the roots and foundations of CM will provide the key for the understanding of QM. Newton's representation of CM has to be replaced with Euler's one, and Euler's method of maxima and minima has to be generalized. Following Helmholtz, the energy law has to be formulated in terms of configurations, including the impossibility of a *perpetuum mobile* and accounting for the relation between energy and extension. Then, it turns out, that CM itself contains the necessary means for going beyond its own

frame. This way, the relationship between CM and non-CM becomes well defined, and the physical content of non-CM is formulated on equal footing with the mathematical method (and vice versa). An example for this is the reformulation of Einstein's criterion in terms of Whittaker's solution method, what allows for the formulation of quantization as selection problem. Moreover, ad hoc assumptions, which may be suggested by experimental results, but are not supported by the axiomatics of CM, can be avoided. The wave and particle aspects can be *obtained* from the time-dependent Schrödinger equation and its solutions (Enders and Suisky, Einführung in die Quantenfeldtheorie des Festkörpers, manuscript in preparation). The classical path in phase space is replaced with the wave functions in space and momentum representations. The wave functions take also the role of the initial conditions, which "are not free, but also have to obey certain laws" (Einstein, 1923).

The dynamics in space and in momentum space is treated *in parallel*. As a consequence, the Schrödinger equation in momentum representation is obtained at once with the one in location representation. This, too, enables one to keep maximum contact to CM and to explain, why QM is a nonclassical mechanics of *conservative* systems, where the *classical* potential and kinetic energy functions and, consequently, the *classical* Lagrange and Hamilton functions still apply. This includes a natural explanation of "the peculiar significance of the energy in quantum mechanics" (Weyl, 1950).

Moreover, the approach presented here allows for *deriving* rather than postulating fundamental properties of *many-body* quantum systems. The *(stationary) states* of systems of equal particles, both classical and quantum ones, are invariant against permutations of the *labels* of the particles involved, cf. (Bach, 1997). As a consequence, the knowledge of only their conserved quantities (total energy, total momentum, *etc*) does *not* allow for distinguishing these particles, neither in quantum, nor in classical systems. The permutation (anti-)symmetry of the wavefunction is a consequence of this permutation invariance of the *states*. Indeed, according to Einstein's derivation of Planck's distribution (Einstein, 1907), the differences between the classical and the quantum distribution laws results from the difference between the *energetic spectrum* of a *single* classical (continuous spectrum) and a *single* quantum oscillator (discrete spectrum), whereas the (in)distinguishability of particles (systems) involved plays no role in Einstein's derivation. Nevertheless, while equal classical bodies are always distinguishable through their location, equal quantum particles are indistinguishable within a common system, because it is impossible to assign individual properties to them (it is the latter point, which avoids the collision with the principle of abstract identity). In this light, Gibbs' paradox seems to suggest, that not the (in)distinguishability of particles is important for the statistical properties of systems, but that of their states.

For the quantization of fields, finally, our approach yields an explanation for the fact, that, within the method of normal-mode expansion, only the temporal,

but not the spatial part of the field variables is concerned, (cf. Schleich, 2001). Indeed, only those variables are subject to the quantization procedure, the possible values of which are, in the classical case of (stationary) states, restricted by the energy law. Thus, the spatial extension of the normal modes being fixed by the boundary conditions is *not* subject to quantization. The classical field energy (density) is determined by the normal-mode coordinates (the amplitudes of the normal modes) and, thus, limits these. As a consequence, the time-depending expansion coefficients in the normal-mode expansion are quantized. When formulating this expansion such, that these expansion coefficients get the dimension of length, their quantization can be performed in complete analogy to that of the harmonic oscillator, *without* invoking additional assumptions or new constants (Enders and Suisky, Einführung in die Quantenfeldtheorie des Festkörpers, manuscript in preparation). Moreover, one could try to quantize a field in the space spanned by independent dynamical field variables, e.g., electric and magnetic field strengths, $\mathbf{E}$ and $\mathbf{H}$. The wave functions would be, say, $\psi(\mathbf{E})$ and $\phi(\mathbf{H})$ rather than $\psi(\mathbf{x})$ and $\phi(\mathbf{p})$. This could separate the quantization problem from the spatial and temporal field distributions and, thus, simplify the realization of Einstein's imagination of a "spatially granular" (Einstein, 1905) structure of the electromagnetic field.

ACKNOWLEDGMENTS

We feel highly indebted to Dr H. Hecht for numerous discussions on the Classical Mechanics in seventeenth and eighteenth centuries. We thank Dr D. B. Fairlie, Prof Mattone, Prof G. Mann, Prof H. Paul, Dr Th. Pöschel, Prof J. Schröter and Prof J. Schnakenberg for helpful discussions. We thank Prof B. Falkenburg for bringing Prof P. Mittelstaedt's analysis to our attention and Prof W. Nolting for his continuous interest and support. Early stages of this work were supported by the Deutsche Akademie der Naturforscher Leopoldina, Prof Th. Elsässer and Prof. E. Siegmund.

REFERENCES

Abramowitz, M. and Stegun, I. A. (eds.). (1964). *Handbook of Mathematical Functions*, NBS, Washington.

d'Alembert, J. L. (1743). *Traité de dynamique*, David l'aine Libraire, Paris.

Bach, A. (1997). *Indistinguishable Classical Particles* (Lecture Notes in Physics, m44), Springer, Berlin, Chap. 1.

Bohr, N. (1913). On the constitution of atoms and molecules. *Philosophical Magazine* **26**, 1ff.

de Broglie, L. (1925). Recherches sur la théorie des quanta. *Annales de Physique* **3**(10), 22.

Cracknell, A. P. (1967). *Applied Group Theory*, Pergamon Press, Oxford, Section 2.10.

Dirac, P. A. M. (1980). Why we believe in the Einstein Theory. In *Symmetries in Science*, B. Gruber, ed., Plenum, New York.

Dyson, F. J. (1949). The radiation theories of Tomonaga, Schwinger, and Feynman. *Physical. Review* **75**, 486–502.

Einstein, A. (1905). Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt. *Annals of Physics* **17**, 132–148.

Einstein, A. (1907). Die Plancksche Theorie der Strahlung und die Theorie der spezifischen Wärme. *Annals of Physics* **22**, 180ff.

Einstein, A. (1923). Bietet die Feldtheorie Möglichkeiten für die Lösung des Quantenproblems? *Sitzungsberichte der Preussischen Akademie der Wissenschaften, physikalisch-mathematische Klasse* (issue December 13) **XXXIII**, pp. 359ff.

Einstein, A. (1977). *Grundzüge der Relativitätstheorie*, Akademie-Verlag, Berlin, p. 132.

Enders, P. (1996). Huygens' principle and the modelling of propagation. *European Journal of Physics* **17**, 226–235.

Enders, P. and Suisky, D. (2005). Über das Auswahlproblem in der klassischen Mechanik und in der Quantenmechanik. *Nova Acta Leopoldina* **18**(Suppl), 13–17.

Euler, L. (1912). *Leonardi Euleri Opera Omnia sub auspiciis Societatis Scientarium Naturalium Helveticae*, Ser. II, Vol. 1, P. Stäckel, ed., Teubner, Leipzig, Zurich.

Euler, L. (1957a). *Opera Omnia*, Ser. II, Vol. 5, O. J. Fleckenstein, ed., Orell Füssli, Zurich.

Euler, L. (1957b). *Opera Omnia*, Ser. II, Vol. 5, O. J. Fleckenstein, ed., Orell Füssli, Zurich.

Euler, L. (1926). *Opera Omnia*, Ser. III, Vol. 1, E. Bernoulli and R. Bernoulli, eds., Teubner, Leipzig.

Euler, L. (1960). *Opera Omnia*, Ser. III, Vols. 11 and 12, A. Speiser, ed., Orell Füssli, Zurich.

Faraggi, A. E. and Matone, M. (1998). The equivalence postulate of quantum mechanics. *arXiv*:hep-th/9809127.

Feynman, R. P. (1949). Space-time approach to quantum electrodynamics. *Physical Review* **76**, 769ff.

Heisenberg, W. (1925). Über quantenmechanische Umdeutung kinematischer und mechanischer Beziehungen. *Zeitschrift fur Physik* **XXXIII**, 879ff.

Heisenberg, W. (1977). *Physik und Philosophie*, Ullstein, Frankfurt, pp. 15–27.

Helmholtz, H. (1847). *Über die Erhaltung der Kraft*, Reimer, Berlin.

Helmholtz, H. von (1911). *Vorlesungen über die Dynamik discreter Massenpunkte*, 2nd edn., Barth, Leipzig.

Landau, L. D. and Lifschitz, E. M. (1959). *Course of Theoretical Physics. Vol. 3: Quantum Mechanics*, Pergamon, London, p. 6.

Leibniz, G. W. (1982). *Specimen Dynamicum*, Meiner, Hamburg.

Lübbig, H. (1999). Das Wirkungsprinzip von Maupertuis und Feynmans Wegintegral der Quantenphase. In *Pierre Louis Moreau de Maupertuis. Eine Bilanz nach 300 Jahren*, H. Hecht, ed., Verlag, Berlin, p. 505.

Malebranche, N. (1906). Looking for the truth. In *Collected Works* (St. Petersburg), Vol.2, p. 426 (in Russian); cited after Grigorjan, A. T. (1985). The initial stage of the development of classical mechanics. In *Investigations on the History of Physics and Mechanics 1985*, Nauka, Moscow, pp. 176–223, in part., p. 193 (in Russian).

Maupertuis, P. L. M. de (1768). *Œuvres de Maupertuis*, Vol. IV, Bruyset, Lyon, p. 44.

Messiah, A. (1999). *Quantum Mechanics*, Dover, New York, Section I.15.

Mittelstaedt, P. (1995). *Klassische Mechanik*, BI –Taschenbücher, Mannheim.

Pauli, W. (1926). Quantentheorie. In *Handbuch der Physik*, Vol. 23, Section 6, H. Geiger and K. Scheel,eds., Springer, Berlin, pp. 1–278.

Pauli, W. (1973). *Wave Mechanics* (Pauli Lectures of Physics, Vol. 5), Ch. P. Enz, ed., MIT Press, Cambridge, MA, Section 35.

Planck, M. (1900). Zur Theorie des Gesetzes der Energieverteilung im Normalspektrum. *Verhandlungen der Deutschen Physikalischen Gesellschaft* **2**, 237ff.

Schleich, W. P. (2001). *Quantum Optics in Phase Space*, Wiley-VCH, Berlin, p. 282.

Schrödinger, E. (1927). *Abhandlungen zur Wellenmechanik*, Barth, Leipzig.

Schrödinger, E. (1933). The fundamental idea of wave mechanics. (Nobel award lecture, 1933). in: G. Ekspong, ed., Nobel Lectures in Physics Vol 2 1922–1941, World Scientific, Singapore, 1998, pp. 305–316.

Suisky, D. and Enders, P. (2001). Leibniz's Foundation of Mechanics and the Development of 18th Century Mechanics initiated by Euler. In *Proc. VII. Intern. Leibniz Congress*, H. Poser, ed., Berlin-Verlag, Berlin, p. 1247.

Suisky, D. and Enders, P. (2005). Dynamical Derivation of the Lorentz Transformation, German Physical Society Annual Meeting, Berlin 2005, Poster GR18.1.

Weizsäcker, C. F. V. (2002). *Aufbau der Physik*, 4th edn., DTV, München, p. 235.

Weyl, H. (1950). *The Theory of Groups in Quantum Mechanics*, Dover, New York, Section II.8.

Whittaker, E. T. and Watson, G. N. (1927). *A Course of Modern Analysis*, 4th ed., Cambridge University Press, Cambridge.